Global Stability of Internet Congestion Controllers With Heterogeneous Delays

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***Abstract*—In this paper, we study the problem of designing globally stable, scalable congestion control algorithms for the Internet. Prior work has primarily used linear stability as the criterion for such a design. Global stability has been studied only for single node, single source problems. Here, we obtain conditions for a general topology network accessed by sources with heterogeneous delays. We obtain a sufficient condition for global stability in terms of the increase/decrease parameters of the congestion control algorithm and the price functions used at the links.**

***Index Terms*—Congestion control, delay system, global stability.**

I. INTRODUCTION

DESIGN of congestion control algorithms for the Internet has received much attention since the work of Kelly *et al.* [11]. Lyapunov techniques were used to analyze the stability property of congestion control algorithms in the absence of delay. The goal of the delay-free analysis was to show that the congestion controllers asymptotically led to fair resource allocation [2], [11], [14], [18], [23], [31], [32]. However, such techniques do not provide insight into how congestion control parameters should be chosen in the presence of feedback delays. A series of papers provided such design guidelines by considering a linearized system and using frequency-domain techniques: first for a single link [8], [12], [13], [21] and then for general network topologies with an arbitrary number of sources and heterogeneous delays [10], [14]–[16], [20], [24], [25], [27]–[29], [34]; see [26] for a comprehensive survey. A significant open challenge is to verify whether the design criteria obtained from such linear analysis ensures global stability or, at least, whether they ensure convergence from a large region of attraction around the equilibrium. Using Razumikhin's theorem, global stability and region of attraction results were obtained in [4] and [7] for the case of congestion managements mechanisms with a dynamic source algorithm and static link law, the so-called primal algorithms. Analogous results were obtained for the case of static source and dynamics algorithms, the so-called dual algorithms, in [30]. The extension of these to the network case has proved to be very difficult, except in the case of very small feedback delays [1] or with other restrictions [5].

In this paper, we study the global stability of Internet congestion controllers following the work of [4]. The key idea in [4] is to first show that the source rates are both upper and lower bounded and then use these bounds in Razumikhin's theorem to derive conditions for global stability. However, a stumbling block in extending the results in [4] to a general network is the difficulty in obtaining reasonable bounds on the source rates and in finding an appropriate Lyapunov–Razumikhin function. In this paper, we take a significant step in this direction by finding a Lyapunov–Razumikhin function that provides global stability conditions for a general topology network with heterogeneous delays.

The global stability condition derived in this paper is delay-independent and is given in terms of the increase/decrease parameters and a parameter of the price function. When the condition holds, the network is globally stable for all values of fixed communication delays and controller gains. It is different from most prior works, where the conditions are given in term of the gains and the delays. Since our global stability condition is delay-independent, the network is robust to the delays and the gains used by users in the network. On the other hand, our stability condition restricts the possible choices for the utility functions and the price functions, whereas stability conditions like [28] and [34] work for general utility functions. Characterizing the stability region when our condition is violated, but the local stability condition still holds, is still an open problem. Our simulation results in Section VI indicate that the region of attraction could be large under such a scenario.

For our purpose, we consider a version of the scalable TCP algorithm suggested in [28] and [34]. For this congestion control algorithm, we show that one can obtain conditions for global stability that relate the parameters of the congestion algorithm to the parameters of the price functions used at the links of the network. Next, we consider a two-phase algorithm, with a slow-start phase followed by a congestion-avoidance phase, as in today's version of TCP-Reno. We will show that a three-phase approximation of this two-phase algorithm is still globally, asymptotically stable under the same conditions on the congestion control parameters. The remainder of the paper is organized as follows. In Section II, we present the mathematical preliminaries required to establish our results. In Section III, we present our system model and derive conditions for global stability. In Section IV, we discuss the relationship between our stability conditions and stability conditions that had been obtained earlier using linear analysis. In Section V, we study the algorithm with the inclusion of a slow-start phase. In Section VI, we validate our results with simulations. Finally, we provide the concluding remarks in Section VII.

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II. Preliminaries

In this section, we will present some basic definitions and the Razumikhin's theorem that will be used to prove our main result.

In the sequel, given an interval $\mathbb{I}$ of the real line $\mathbb{R}$, we will use $C(\mathbb{I}, \mathbb{R}^n)$ to denote the set of continuous functions mapping $\mathbb{I}$ to $\mathbb{R}^n$, equipped with the sup norm. We define the following important object.

Definition 1: Suppose $d > 0$ and that S is a bounded subset of $C = C([-d, 0], \mathbb{R}^n)$. Given $x \in C([-d, \infty), \mathbb{R}^n)$ and any $t \geq 0$, let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-d, 0]$. Given a locally Lipschitz mapping $f : S \to \mathbb{R}^n$, we say the equation

$$\dot{x}(t) = f(x_t)$$

defined for $t \geq 0$ is a retarded differential equation (RFDE) with domain S. ◇

Given an initial condition $\phi \in S$, we say that x is a corresponding solution if $x_0 = \phi$ and $x_t \in S$ for all $t \geq 0$. If the RFDE has a unique solution for all initial conditions in S, then it is said to be well-posed.

The version of Razumikhin's theorem that will play a central role in this paper is now stated; it follows directly from [19, Theorem 7.3.1].

Theorem 2: Suppose that the RFDE is well posed. If there exists a continuous function $W : \mathbb{R}^n \to [0, \infty)$ which takes the value zero only at $\hat{x}$ and which satisfies the inequality

$$\lim_{a \to 0^+} \sup \frac{1}{a} \left\{ W\left(x(t+a)\right) - W\left(x(t)\right) \right\} < 0$$

for each t satisfying $\alpha^2 W(x(t)) > \max_{r \in [t-d, t]} W(x(r))$ and some $\alpha^2 > 1$, then every bounded solution $x(t)$ converges asymptotically to $\hat{x}$. ◇

Consider a function $V(t) = W(x(t))$. To ensure global asymptotic stability, Lyapunov theory requires $V(t)$ to decrease for all time t till the equilibrium is reached. Razumikhin's theorem relaxes this condition and only requires $V(t)$ to decrease at every t such that

$$\alpha^2 V(t) > \max_{t-d \leq r \leq t} V(r). \tag{1}$$

Thus, as shown in Fig. 1, the Lyapunov function can increase occasionally as long it decreases whenever it is larger than $1/\alpha^2$ times its maximum value over a time interval of duration d. This completes the preliminaries.

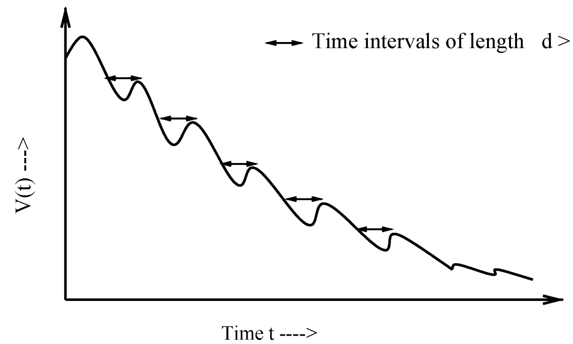


Fig. 1. Plot illustrating the behavior of $V(t)$ under the conditions of Razumikhin's theorem.

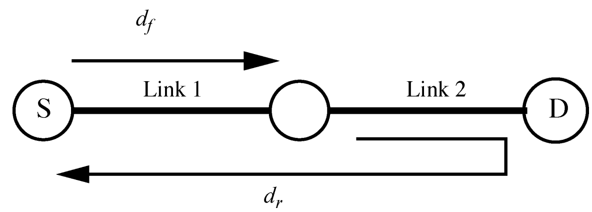


Fig. 2. Network with two links.

III. Global Stability

We consider a general topology network consisting of an arbitrary number of sources and links. Each source is assumed to use a fixed route (a collection of links) from its origin to its destination and, therefore, we will use the same notation for both a source and its route. In other words, we may use the index i to denote either source i or the set of links used by source i. Let $d_f(i, l)$ be the forward delay from source i to link l and $d_r(i, l)$ be the reverse delay from link l to source i. An illustration of d_f and d_r is provided in Fig. 2. Denote by $T_i = d_f(i, l) + d_r(i, l)$ the round-trip time (RTT) for source i.

We consider the following TCP-like congestion control algorithm suggested in [28] and [34]:

$$\dot{x}_i(t) = \kappa_i x_i(t - T_i) \left(\frac{a_i}{x_i^{n_i}(t)} - b_i x_i^{m_i}(t) q_i(t) \right) \tag{2}$$

where

$$q_i(t) = \sum_{l \in i} p_l\left(t - d_r(i, l)\right) \tag{3}$$

$$p_l(t) = f_l\left(y_l(t)\right) \tag{4}$$

$$f_l\left(y_l(t)\right) = \left(\frac{y_l(t)}{c_l} \right)^{h_l} \tag{5}$$

and

$$y_l(t) = \sum_{k: l \in k} x_k\left(t - d_f(k, l)\right).$$

Here, a_i, b_i, and h_l are positive real numbers, and m_i and n_i are real numbers that satisfy $m_i + n_i > 0$. In the above set of equations, x_i is the rate at which source i transmits data, y_l is the arrival rate at link l, p_l is the price of link l, q_i is the price of source i's route, and $f_l(y)$ is the function of the link arrival rate which is used to compute p_l. The price of a route is simply the sum of the prices of the links along its path. Also, define the quantities

$$d := \max_{i,k,l} \left(d_r(i, l) + d_f(k, l)\right)$$
$$H_i := \max_{l \in i} h_l$$

which will be useful throughout the paper.

Associated with the model is the initial condition given by continuous functions ϕ_i on $[-d, 0]$; when each of these functions is strictly positive, we say the network model has a *nonzero initial condition*. We will restrict our attention to this situation. It is possible to show that there is a unique positive equilibrium point for the above set of equations; namely there is only one positive constant solution $x(t) = \hat{x}$; see, for instance, [26, Ch. 3].

The above algorithm can be used to model the increase/decrease behavior of today's versions of TCP (such as Reno and NewReno) as well as versions of TCP that have been suggested for scalable data transmission over very high-speed links and over large RTTs [26]. Our first result shows that, provided the network has a nonzero initial condition, each of the flows $x_j(t)$ is uniformly bounded from above and below, with the lower bound being strictly greater than zero. The result provides explicit bounds in terms of the initial network condition.

Proposition 3: Suppose the network model in (2) has a nonzero initial condition $m_j + n_j > H_j$ for all j and that the constant γ satisfies

$$\gamma \geq \max_{r\in[-d,0]} \left|\log \frac{\phi_j(r)}{\hat{x}_j}\right| \text{ for each } j.$$

Then, each of the flows x_j satisfies

$$\hat{x}_j e^{-\gamma} \leq x_j(t) \leq \hat{x}_j e^{\gamma}$$

for all time $t \geq -d$.

Proof: We will prove the desired inequality by contradiction. Note from the initial condition that the bound on $x_j(t)$ holds in the interval $t \in [-d, 0]$ for all j.

Suppose that some flow x_i violates the inequality on the interval $(0, \infty)$, then, by continuity of the functions x_j and their derivatives, there must exist a time $t \geq 0$ such that:

(a) for each j, the inequality $\hat{x}_j e^{-\gamma} \leq x_j(r) \leq \hat{x}_j e^{\gamma}$ holds on $[-d, t]$.

(b) at time t, one of following conditions holds

$$\gamma = \log \frac{x_i(t)}{\hat{x}_i} \text{ and } \dot{x}_i(t) > 0 \tag{6}$$

$$\gamma = -\log \frac{x_i(t)}{\hat{x}_i} \text{ and } \dot{x}_i(t) < 0. \tag{7}$$

We will show that this leads to a contradiction.

Recall that the arrival rate at any link l is defined as

$$y_l(t) = \sum_{k:l\in k} x_k\left(t - d_f(k,l)\right).$$

Then, using condition (a) above, we have that

$$\left(\frac{\hat{y}_l e^{-\gamma}}{c_l}\right)^{h_l} \leq \left(\frac{y_l\left(t - d_r(i,l)\right)}{c_l}\right)^{h_l} \leq \left(\frac{\hat{y}_l e^{\gamma}}{c_l}\right)^{h_l}$$

and thus

$$\hat{q}_i e^{-\gamma H_i} \leq q_i(t) \leq \hat{q}_i e^{\gamma H_i}$$

where $\hat{q}_i = \sum_{l\in i}(\hat{y}_l/c_l)^{h_l}$. Now suppose that (6) holds, then $x_i(t) = \hat{x}_i e^{\gamma}$ and $\dot{x}_i > 0$, but from the system equation and the above lower bound on $q_i(t)$ we have

$$\begin{aligned}\dot{x}_i(t) &= \kappa_i x_i(t - T_i)\left(\frac{a_i}{x_i^{n_i}(t)} - b_i x_i^{m_i}(t) q_i(t)\right)\\ &= \kappa_i x_i(t - T_i)\left(\frac{a_i}{(e^{\gamma}\hat{x}_i)^{n_i}} - b_i (e^{\gamma}\hat{x}_i)^{m_i} q_i(t)\right)\\ &< \kappa_i x_i(t - T_i)\left(\frac{a_i}{(e^{\gamma}\hat{x}_i)^{n_i}} - b_i (e^{\gamma}\hat{x}_i)^{m_i} e^{-\gamma H_i}\hat{q}_i\right)\\ &< 0\end{aligned}$$

where the last inequality is derived from the equilibrium condition $\hat{q}_i = (a_i/b_i)\hat{x}_i^{-m_i-n_i}$. Thus, (6) cannot hold. Similarly, we can show that a contradiction arises when (7) holds. Hence, we conclude that no such time t exists and thus the sought bounds must be valid for all flows for all time. ■

From Proposition 3, we have $x(t) > 0$ for all t. So, we can define the functions

$$W_j(t) = \frac{1}{2}\left(\log x_j(t) - \log \hat{x}_j\right)^2 \tag{8}$$

which, by the above result, are well-defined provided that the network model has a nonzero initial condition. Also, define the function

$$W(t) = \max_j W_j(t). \tag{9}$$

Recall that, to apply Razumikhin's theorem, we are interested in those time instants t at which

$$\alpha^2 W(t) > \max_{t-d\leq r\leq t} W(r) \tag{10}$$

for some $\alpha > 1$. The next lemma shows that, if the Lyapunov function W defined by (8) and (9) satisfies the condition in (10) at some time instant t, then this naturally imposes upper and lower bounds on the functions x_j over the interval $\in [t - d, t]$.

Lemma 4: Suppose the network model has a nonzero initial condition that at time t the Lyapunov function W satisfies the condition in (10) and that index i is such that

$$W(t) = W_i(t).$$

Then

$$B_i^{-\beta}\hat{x}_j < x_j(r) < B_i^{\beta}\hat{x}_j \tag{11}$$

for all $r \in [t - d, t]$, where $B_i = x_i(t)/\hat{x}_i$ and $\beta = \operatorname{sgn}(\log B_i)\alpha$.

Proof: First, note that, since $W_i(t) \neq 0$, we have $B_i \neq 1$. By the definition on $W_i(t)$, we have for each index j and $r \in [t - d, t]$ that

$$\alpha^2(\log B_i)^2 = \alpha^2 W(t) > W_j(r) = \frac{1}{2}\left(\log \frac{x_j(r)}{\hat{x}_j}\right)^2.$$

Note that the above inequality can be rewritten as

$$(\log B_i^{\alpha})^2 > \left(\log \frac{x_j(r)}{\hat{x}_j}\right)^2$$

which implies that

$$-\text{sgn}(\log B_i)\log B_i^{\alpha} < \log \frac{x_j(r)}{\hat{x}_j} < \text{sgn}(\log B_i)\log B_i^{\alpha}.$$

From this, we get the inequalities

$$\hat{x}_j B_i^{-\beta} < x_j(r) < \hat{x}_j B_i^{\beta}.$$

■

The next lemma then shows that the route prices are also upper and lower bounded as a consequence of the previous lemma.

Lemma 5: Under the conditions of Lemma 4, the price of route i at time t is bounded as given by the following expression:

$$\frac{a_i}{b_i} B_i^{-H_i\beta} \hat{x}_i^{-m_i-n_i} < q_i(t) < \frac{a_i}{b_i} B_i^{H_i\beta} \hat{x}_i^{-m_i-n_i}.$$

Proof: Similar to the bounds on q used in Proposition 3, we have

$$\hat{q}_i B_i^{-\beta H_i} < q_i(t) < \hat{q}_i B_i^{\beta H_i}.$$

Furthermore, because $\hat{q}_i = (a_i/b_i)\hat{x}_i^{-m_i-n_i}$, we can conclude that

$$\frac{a_i}{b_i} B_i^{-H_i\beta} \hat{x}_i^{-m_i-n_i} < q_i(t) < \frac{a_i}{b_i} B_i^{H_i\beta} \hat{x}_i^{-m_i-n_i}.$$

■

Corollary 6: If $m_i + n_i > H_i$, the supposition of Lemma 4 holds and $\alpha = (m_i + n_i + H_i)/2H_i$, then $\dot{W}_i(t) < 0$.

Proof: The derivative $\dot{W}_i(t)$ is given by

$$\dot{W}_i(t) = \frac{\dot{x}_i(t)}{x_i(t)} \log\left(\frac{x_i(t)}{\hat{x}_i}\right) = \frac{\dot{x}_i(t)}{x_i(t)} \log B_i$$

and, thus, to prove the result, we need to show that $\dot{x}_i(t)\log B_i < 0$. From (2), we see that

$$\dot{x}_i(t) = \kappa_i \frac{x_i(t-T_i)}{x_i^{n_i}(t)} \left(a_i - b_i q_i(t) x^{m_i+n_i}(t)\right).$$

Since $x_i(t-T_i)$ and $x_i(t)$ are both positive, it is sufficient to show that

$$\left(a_i - b_i q_i(t) x^{m_i+n_i}(t)\right) \log B_i < 0. \tag{12}$$

We can show this using Lemma 5 by considering the following two cases. Suppose first that $B_i > 1$, then we have to show that

$$q_i(t) x_i^{m_i+n_i}(t) > \frac{a_i}{b_i}.$$

From the lower bound in Lemma 5, we have

$$\begin{aligned} q_i(t) x_i^{m_i+n_i}(t) &> \frac{a_i}{b_i} B_i^{-H_i\alpha} \hat{x}_i^{-m_i-n_i} x_i^{m_i+n_i}(t) \\ &= \frac{a_i}{b_i} B_i^{-H_i\alpha+m_i+n_i} \end{aligned}$$

which is greater than a_i/b_i since $\alpha H_i < m_i + n_i$. Next, suppose that $B_i < 1$. Then, we have to show that

$$q_i(t) x_i^{m_i+n_i}(t) < \frac{a_i}{b_i}.$$

From the upper bound in Lemma 5

$$\begin{aligned} q_i(t) x_i^{m_i+n_i}(t) &< \frac{a_i}{b_i} B_i^{-H_i\alpha} \hat{x}_i^{-m_i-n_i} x_i^{m_i+n_i}(t) \\ &= \frac{a_i}{b_i} B_i^{-H_i\alpha+m_i+n_i} \end{aligned}$$

which is now less than a_i/b_i since $\alpha H_i < m_i + n_i$. ■

This brings us to the main result of the paper.

Theorem 7: If $m_i + n_i > H_i > 0$, then the network model in (2) is globally asymptotically stable.

Proof: By the hypothesis $\alpha = ((m_i + n_i + H_i)/2H_i) > 1$ and furthermore, in Proposition 3, we have shown that $x(t)$ is bounded. Now, invoking Razumikhin's theorem, with $W(t)$ and $W_j(t)$ as defined above, it is enough to prove for every $t \geq 0$ that

$$\limsup_{a\to 0^+} \frac{1}{a}\left\{W(t+a) - W(t)\right\} < 0 \tag{13}$$

whenever $\alpha^2 W(t) > \max_{t-d\leq r\leq t} W(r)$. Fix any such t and let $\mathcal{I}$ denote the set of indexes i satisfying $W_i(t) = W(t)$. Since each function $W_j(t)$ is continuous, there exists a neighborhood of zero such that for every a in this neighborhood

$$\max_j W_j(t+a) = \max_{i\in\mathcal{I}} W_i(t+a) \text{ holds.}$$

Thus, for every positive a in this neighborhood

$$\frac{1}{a}\left\{W(t+a) - W(t)\right\} = \max_{i\in\mathcal{I}} \frac{1}{a}\left\{W_i(t+a) - W_i(t)\right\}.$$

By Lemma 6, each $\dot{W}_i(t) < 0$, and so there exists $\varepsilon > 0$ such that

$$\max_{i\in\mathcal{I}} \frac{1}{a}\left\{W_i(t+a) - W_i(t)\right\} \leq -\varepsilon$$

for all $a > 0$ sufficiently near to zero. This implies (13) as required. ■

Control law (2) is a TCP-like algorithm. From [11] and [26], we know that the congestion control algorithm can be interpreted as a distributed resource allocation scheme: suppose that each user has a monotonically increasing and concave utility function $U_i(x_i)$ and the goal of the congestion control algorithm is to allocate the resources so that the equilibrium point $\hat{\mathbf{x}}$ solves the following optimization problem:

$$\max \sum_i U_i(x_i) - \sum_l \int_0^{y_l} p_l(y)dy. \quad (14)$$

Under appropriate concavity assumptions, the solution of the above problem can be obtained by solving

$$U_i'(x_i) = q_i. \quad (15)$$

Comparing (15) to the equilibrium point of (2), we get

$$U_i'(x_i) = \frac{a_i}{b_i}\frac{1}{x_i^{n_i+m_i}}.$$

Note that this belongs to the general class of $\alpha-$ fair utility functions introduced in [22]. Thus, our condition in Theorem 7 places a restriction on the utility function and price function.

Remark: A more commonly studied form of the controller is

$$\dot{x}_i(t) = \kappa_i x_i(t-T_i)\left(1 - \frac{q_i(t)}{U_i'(x_i(t))}\right).$$

Our results continue to apply to this algorithm as well. The sufficient condition for global stability can be stated as follows: the network is globally, asymptotically stable if for all $\sigma > 0$

$$\left|\log \frac{U_i'(\sigma\hat{x}_i)}{U_i'(\hat{x}_i)}\right| > H_i|\log\sigma|.$$

IV. Local Stability

The global stability condition in the previous section imposes no constraint on the gains $\{\kappa_i\}$, but rather imposes a condition on the increase/decrease parameters of TCP and the price function. This is in contrast to the results in [28] and [34], where κ_i is required to be smaller than some constant times the RTT on route i. We now show that the condition derived in [28] and [34] is more conservative than what is necessary to ensure locally stability when $m_i + n_i > H_i$. Consider the congestion controller

$$\dot{x}_i(t) = \kappa_i x_i(t-T_i)\left(\frac{a_i}{x_i^{m_i}(t)} - b_i q_i(t) x_i^{n_i}(t)\right). \quad (16)$$

After linearizing the system around the equilibrium point (suppose $\delta x_i(t) = x_i(t) - \hat{x}_i$ and $\delta q_i(t) = q_i(t) - \hat{q}_i$) and then taking the Laplace transforms, we get

$$s\delta x_i(s) + x_i(0) = \kappa_i\hat{x}_i\left(-m_i\frac{a_i}{\hat{x}_i^{m_i+1}}\delta x_i(s) - b_i\hat{x}_i^{n_i}\delta q_i(s) \right.$$
$$\left. - n_i b_i \hat{q}_i \hat{x}_i^{n_i-1}\delta x_i(s)\right).$$

Defining the Laplace routing matrix $R(s)$ as follows [28], [34]:

$$R_{li}(s) = \begin{cases} e^{-sd_f(i,l)}, & \text{if source } i \text{ uses link } l \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\delta q_i(s) = e^{-sT_i}\sum_l R_{il}^T(-s)\left(h_l\frac{\hat{p}_l}{\hat{y}_l}\sum_j R_{lj}(s)\delta x_j(s)\right)$$

and

$$\delta x(0) = \left(\mathrm{diag}\left(s + \kappa_i a_i m_i \hat{x}_i^{-m_i} + n_i b_i \hat{q}_i x_i^{n_i}\right)\right)\delta x(s)$$
$$+ \left(\mathrm{diag}\left(\kappa_i b_i \hat{x}_i^{n_i+1} e^{-sT_i}\right) R^T(-s)\mathrm{diag}\left(\frac{h_l\hat{p}_l}{\hat{y}_l}\right)R(s)\right)\delta x(s).$$

Further, since $\mathrm{diag}(H_i)R^T(-s)\mathrm{diag}(1/H_i) = R^T(-s)$ and $a_i/\hat{x}_i^{m_i} = b_i\hat{x}_i^{n_i}\hat{q}_i$, we can obtain

$$\delta x(0) = \left(\mathrm{diag}\left(s + \kappa_i a_i \hat{x}_i^{-m_i}(m_i+n_i)\right)\right)\delta x(s)$$
$$+ \left(\mathrm{diag}\left(\kappa_i H_i a_i \hat{x}_i^{-m_i} e^{-sT_i}\frac{\hat{x}_i}{\hat{q}_i}\right)\right.$$
$$\left. \times R^T(-s)\mathrm{diag}\left(\frac{h_l\hat{p}_l}{H_i\hat{y}_l}\right)R(s)\right)\delta x(s).$$

From basic control theory, the above system is stable if its poles lie in the left half of the complex plane, i.e., the solution to

$$\det\left((s + c_i(m_i+n_i))I + (s + c_i(m_i+n_i))G\right) = 0$$

should have negative real parts, where

$$G = \mathrm{diag}\left(\kappa_i T_i H_i a_i \hat{x}_i^{-m_i}\frac{e^{-sT_i}}{sT_i + \kappa_i T_i a_i \hat{x}_i^{-m_i}(m_i+n_i)}\right)$$
$$\times\, \mathrm{diag}\left(\frac{\hat{x}_i}{\hat{q}_i}\right)R^T(-s)\mathrm{diag}\left(\frac{h_l\hat{p}_l}{H_i\hat{y}_l}\right)R(s)$$
$$c_i = \kappa_i T_i a_i \hat{x}_i^{-m_i}.$$

It is easy to see that $s = -c_i(m_i+n_i)$ cannot be a solution to $\det((s+c_i(m_i+n_i))I + (s+c_i(m_i+n_i))G) = 0$. Therefore, we can equivalently check if the roots of $\det(I+G) = 0$ have negative real parts.

As in [28] and [34], it can be shown using the multivariable Nyquist criterion that the stability condition is equivalent to the following statement: the eigenvalues of $G(j\omega)$ should not encircle the point -1. It has been proved in [28] and [34] that

$$\left|\lambda\left(\mathrm{diag}\left(\frac{\hat{x}_i}{\hat{q}_i}\right)R^T(-s)\mathrm{diag}\left(\frac{\hat{p}_l h_l}{\hat{y}_l H_i}\right)R(s)\right)\right| \leq 1$$

where $\lambda(M)$ is any eigenvalue of M. Furthermore, if $m_i + n_i > H_i$, we have

$$\left|\left(c_i H_i \frac{e^{-sT_i}}{sT_i + c_i(m_i+n_i)}\right)\right| = \left|\left(\frac{e^{-j\omega T_i}}{j\omega\frac{T_i}{c_i H_i} + \frac{m_i+n_i}{H_i}}\right)\right|$$
$$= \frac{1}{\sqrt{\left(\frac{m_i+n_i}{H_i}\right)^2 + \left(\frac{\omega}{c_i H_i}\right)^2}}$$
$$< 1$$

which implies that $|\lambda(G(j\omega))| < 1$, so the network is locally stable when $m_i + n_i > H_i$. Thus, the condition

$$\kappa_i T_i \leq \frac{\pi}{2}\frac{1}{\max_i H_i}$$

given in [28] and [34] is not necessary when $m_i + n_i > H_i$.

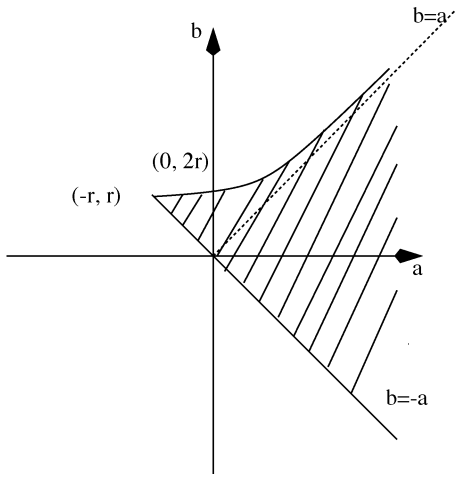


Fig. 3. Exact region of asymptotic stability.

Next, we will use local analysis to show that $m_i + n_i > H_i$ is a very tight condition for a general topology network with heterogeneous delays to be globally asymptotically stable for any positive T_i and κ_i. Considering a single-source, single-link network, the congestion control algorithm is

$$\dot{x}(t) = \kappa x(t-T)\left(\frac{a}{x^m(t)} - bx^n(t)\left(\frac{x(t-T)}{c}\right)^h\right).$$

Linearizing the differential equation near the equilibrium point, we get

$$\dot{z}(t) = -\left(\frac{\kappa ma}{\hat{x}^m} + \frac{\kappa na}{\hat{x}^m}\right)z(t) - \frac{\kappa ha}{\hat{x}^m}z(t-T). \tag{17}$$

Now, defining $A = (m+n)a/\hat{x}^m$ and $B = ha/\hat{x}^m$, the delay differential equation becomes

$$\dot{z}(t) = -\kappa Az(t) - \kappa Bz(t-T). \tag{18}$$

From [6, p. 135], the set of values for A, B, and κ such that (18) is asymptotically stable is given by the shaded area of Fig. 3. Thus, when $A < B$ $(m+n < h)$, we can always find a κ large enough such that $(\kappa A, \kappa B)$ is not in the shaded region. This means that, when $m+n < h$, we can always find a κ such that the delay differential equation (17) is not asymptotically stable. A single-source, single-link network is a special case of a network, so we can conclude that $m_i + n_i > H_i$ is a tight condition for a general topology network with heterogeneous delays to be globally, asymptotically stable for any positive κ_i and T_i.

V. Dynamic Phases and Slow Start

Today's TCP congestion control consists of two phases—a slow-start phase and a congestion-avoidance phase. The slow-start phase begins when the TCP connection is established or congestion is indicated by a timeout. In that phase, the window size increases by 1 for every ACK received until the window size reaches the slow-start threshold (*ssthresh*). The key idea behind using the two-phase dynamics is to use slow start to bring the source rates near the equilibrium point very quickly and then use congestion avoidance to make the system converge to its equilibrium point [9]. The question here is that, although we have shown in Section III that under the condition $m_i + n_i > H_i$ the network is globally, asymptotically stable, will stability still be preserved in the presence of a slow-start mechanism? In this section, we investigate an approximation to this problem. We will first introduce a two-phase discontinuous model that acts like slow start and then approximate it with a three-phase continuous model which we are able to analyze with the Razumikhin–Lyapunov technique above.

First, we suppose the rate $x_i(t)$ and window size $w_i(t)$ satisfy the approximate relation

$$w_i(t) = x_i(t)T_i. \tag{19}$$

Then, substituting for $x_i(t)$ in terms of $w_i(t)$ in (2) gives the following window-based algorithm:

$$\dot{w}_i(t) = \kappa_i w_i(t-T_i)\left(\frac{\tilde{a}_i}{w_i^{n_i}(t)} - \tilde{b}_i w_i^{m_i}(t)q_i(t)\right)$$

where

$$\tilde{a}_i = a_i T_i^{n_i} \tag{20}$$

$$\tilde{b}_i = \frac{b_i}{T_i^{m_i}}. \tag{21}$$

Motivated by TCP's slow-start phase, we first define the two-phase algorithm for the window evolution

$$\frac{\dot{w}_i(t)}{\kappa_i w_i(t-T_i)} = \begin{cases} \frac{\tilde{a}_i}{w_i^{n_i}(t)} - \tilde{b}_i w_i^{m_i}(t)q_i(t), & \text{if } w_i(t) \geq v_i; \\ \frac{\tilde{a}_i}{w_i^{l_i}(t)} - \tilde{b}_i w_i^{m_i}(t)q_i(t), & \text{if } w_i(t) < v_i \end{cases} \tag{22}$$

where $l_i < n_i$ and v_i is the threshold window size at which the algorithm switches from one phase to the other. Furthermore, we assume that $w_i(t) \geq 1$ for all t, which means that the window size is at least 1 when the source has data to transmit. Under this assumption, we have that for all t

$$\frac{\tilde{a}_i}{w_i^{n_i}(t)} < \frac{\tilde{a}_i}{w_i^{l_i}(t)}. \tag{23}$$

Thus, source i is in a fast increase regime when $w_i(t) < v_i$; furthermore, because l_i can be any real number, so we can choose l_i to make the window size increase as fast as we want in this phase. Now we have a well-defined window-based two-phase algorithm. Use (19)–(21), we can derive the corresponding rate-based two-phase algorithm:

$$\dot{x}(t) = \begin{cases} \kappa_i x_i(t-T_i) \\ \quad \times \left(\frac{a_i}{x_i^{n_i}(t)} - b_i x_i^{m_i}(t)q_i(t)\right), & \text{if } x_i(t) \geq \chi_i \\ \kappa_i x_i(t-T_i) \\ \quad \times \left(\frac{\mu_i}{x_i^{l_i}(t)} - b_i x_i^{m_i}(t)q_i(t)\right), & \text{if } x_i(t) < \chi_i \end{cases} \tag{24}$$

where $\mu_i = \tilde{a}_i/T_i^{l_i}$ and $\chi_i = v_i/T_i$. From (23), we have the following inequality:

$$\frac{a_i}{x_i^{n_i}(t)} > \frac{\mu_i}{x_i^{l_i}(t)} \tag{25}$$

for all t. Now, since $l_i < n_i$, source i is in a fast increase regime when $x_i(t) < \chi_i$ and in a slow increase regime when $x_i \geq \chi_i$. The problem with analyzing this algorithm is that $\dot{x}_i(t)$ is not continuous, so we make the following modification to the algorithm:

$$\dot{x}_i(t)=\begin{cases} \kappa_i x_i(t-T_i) \\ \quad \times\left(\frac{a_i}{x_i^{n_i}(t)} - b_i x_i^{m_i}(t) q_i(t)\right), & \text{if } x_i(t)\geq\chi_i \\ \kappa_i x_i(t-T_i) \\ \quad \times\left(\frac{\mu_i(x_i(t))}{x_i^{l_i(x_i(t))}(t)} - b_i x_i^{m_i}(t) q_i(t)\right), & \text{if } x_i(t)<\chi_i \end{cases} \tag{26}$$

where

$$\begin{aligned} &\mu_i\left(x_i(t)\right) \\ &\quad =\begin{cases} \mu_i T_i^{(x_i(t)-\chi_i+\delta)\frac{l_i-n_i}{\delta}}, & \text{if } \chi_i-\delta < x_i(t) < \chi_i \\ \mu_i, & \text{if } x_i(t) \leq \chi_i - \delta \end{cases} \\ &l_i\left(x_i(t)\right) \\ &\quad =\begin{cases} l_i + (x_i(t)-\chi_i+\delta)\frac{n_i-l_i}{\delta}, & \text{if } \chi_i-\delta < x_i(t) < \chi_i \\ l_i, & \text{if } x_i(t) \leq \chi_i - \delta \end{cases} \end{aligned}$$

and δ is a small positive number.

Notice that $a_i = \mu_i T_i^{-(n_i-l_i)}$ and

$$(x_i(t)-\chi_i+\delta)\frac{n_i-l_i}{\delta} = \begin{cases} 0, & \text{if } x_i(t)=\chi_i-\delta \\ n_i-l_i, & \text{if } x_i(t)=\chi_i. \end{cases} \tag{27}$$

Therefore, $\dot{x}_i(t)$ is continuous now, but the dynamics has three phases. Note that, when $\delta \to 0$, the derivative function in (26) converges to the derivative function defined in the two-phase algorithm (24). Also, from inequality (25), we have that, if $x_i(t)T_i \geq 1$ for all t, then

$$\frac{a_i}{x_i^{n_i}(t)} \leq \frac{\mu_i\left(x_i(t)\right)}{x_i^{l_i(x_i(t))}(t)}. \tag{28}$$

Now we have well-defined delay-differential equations, and we can show that, if $m_i+n_i > H_i > 0$ and $\chi_i < \hat{x}_i$ for all i, then the network with three-phase algorithm is globally, asymptotically stable too.

Theorem 8: Suppose $x_i(t)T_i \geq 1$. If $m_i+n_i > H_i > 0$ and $\chi_i < \hat{x}_i$ for all i, then the network defined by (26) is globally, asymptotically stable.

Proof: The idea behind this proof is the same as for the proof of Theorem 7. We will show that

$$\dot{W}_i(t) < 0$$

if $W_i(t)=W(t)$ and $\alpha^2 W(t) > \max_{t-d\leq r\leq t} W(r)$. First, we know that, under this two-phase algorithm, Lemmas 4 and 5 still hold. So we have

$$\frac{a_i}{b_i}B_i^{-H_i\beta}\hat{x}_i^{-m_i-n_i} < q_i(t) < \frac{a_i}{b_i}B_i^{H_i\beta}\hat{x}_i^{-m_i-n_i}.$$

Now let $x_i(t) = B_i\hat{x}_i$ and consider $\dot{W}_i(t)$. First, suppose $B_i\hat{x}_i \geq \chi_i$. In this case, the same argument as for the proof of Theorem 7 applies and we have $W_i(t) < 0$. Next, we consider

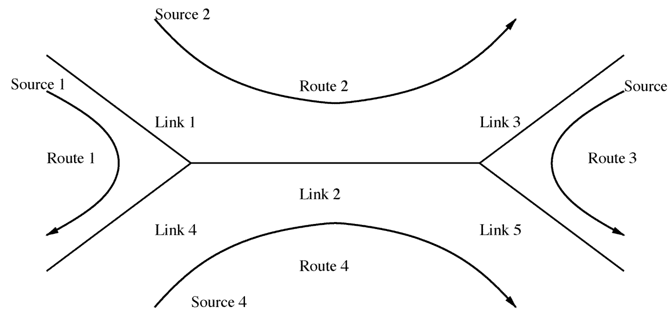


Fig. 4. Network used in the simulation.

TABLE I
MATLAB PROGRAM FOR THE SINGLE-PHASE ALGORITHM

Matlab Algorithm

for $k = 0:1:T/\text{step}$
    for $l = 0:1:L$
        $p_l(k) = \left(\frac{\sum_{l\in j} x_j(k-D_f(i,l))}{c_l}\right)^{h_l}$;
    end
    for $i = 0:1:N$
        $\Delta x_i(k) = \text{step} \times \kappa_i x_i(k-D_i)$
            $\times\left(\frac{w}{x_i^{0.1}(k)} - x_i(k)\sum_{l:l\in i} p_l(k-D_r(i,l))\right)$;
        $x_i(k+1) = x_i(k) + \Delta x_i(k)$;
    end
end

T : The simulation time interval $D_i : D_i = T_i/\text{step}$ step: The simulation step size L : Number of links N : Number of users $D_f(i,l) : d_f(i,l)/\text{step}$ $D_r(i,l) : d_r(i,l)/\text{step}$

$B_i\hat{x}_i < \chi_i$. Under the assumption $\chi_i < \hat{x}_i$, we have $B_i < 1$ and

$$q_i(t)x_i^{m_i+n_i}(t) < \frac{a_i}{b_i}B_i^{-H_i\alpha+m_i+n_i}.$$

Then, based on (28), we have

$$\begin{aligned} \frac{x_i^{l_i(x_i(t))}(t)}{\mu_i\left(x_i(t)\right)} q_i(t) x_i^{m_i+n_i}(t) &\leq \frac{x_i^{n_i}(t)}{a_i} q_i(t) x_i^{m_i+n_i}(t) \\ &< \frac{1}{b_i}B_i^{-H_i\alpha+m_i+n_i} \\ &< \frac{1}{b_i}. \end{aligned}$$

So we have

$$\frac{x_i^{l_i(x_i(t))}(t)}{\mu_i\left(x_i(t)\right)} b_i q_i(t) x_i^{m_i}(t) < 1$$

which implies that $\dot{W}_i(t) < 0$, so the network is globally, asymptotically stable. ■

The slow-start phase is a very important part of today's TCP congestion control algorithm. For the algorithms considered in this paper, if H_i is large, Theorem 7 tells us that we need a large n_i, which means the increase rate of user i will be quite slow and it will take a quite long time for the network to converge to the

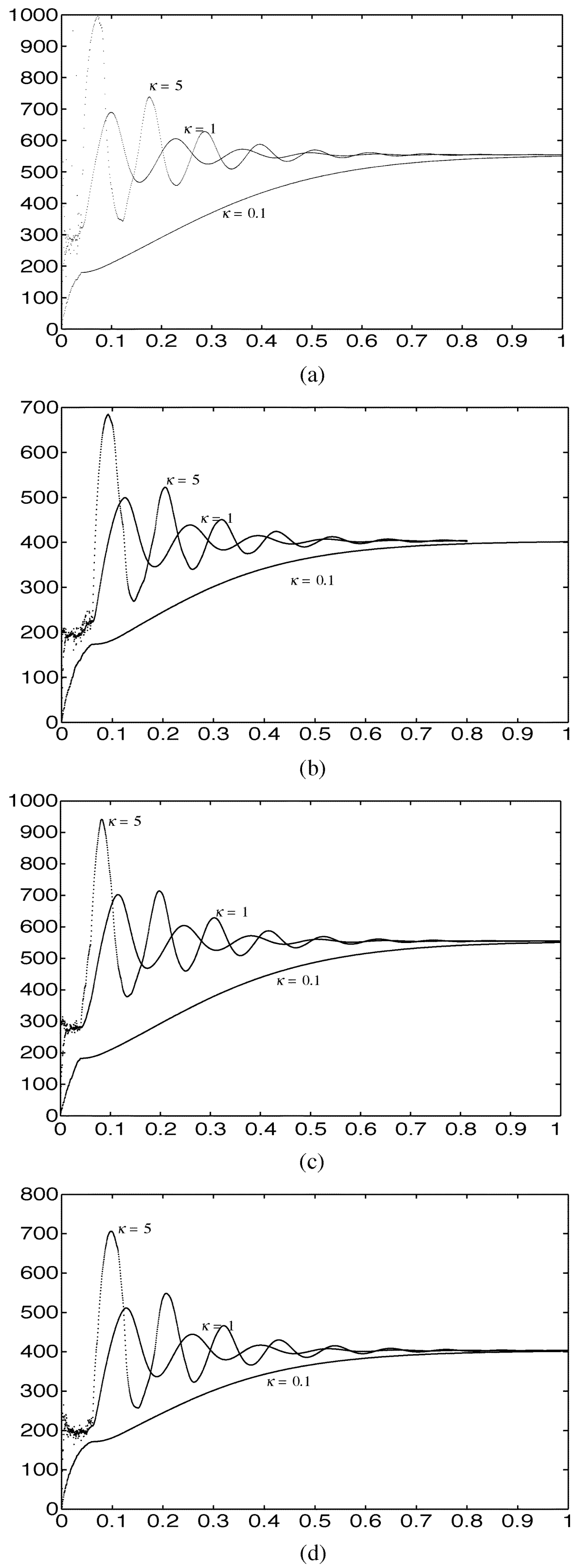


Fig. 5. Rates for $\kappa_i = 0.1, \kappa_i = 1$ and $\kappa_i = 5$. (a) Typical user's rate on route 1; (b) typical user's rate on route 2; (c) typical user's rate on route 3; (d) typical user's rate on route 4.

TABLE II
MATLAB PROGRAM FOR THE TWO-PHASE ALGORITHM

Matlab Algorithm — Two Dynamics
for $i = 0:1:N$
if $x_i(k) < 300$
$n_i = -0.1;$
end
else
$n_i = 0.1;$
end
$\Delta x_i(k) = \text{step} \times \kappa_i x_i(k - D_i)$
$\times \left(\frac{w}{x_i^{n_i}(k)} - x_i(k) \sum_{l:l \in i} p_l(k - D_r(i,l)) \right);$
$x_i(k+1) = x_i(k) + \Delta x_i(k);$
end

equilibrium point. But, if we use the two-phase algorithm, we can drive the network near the equilibrium point very quickly and then stabilize it. Thus, we can guarantee stability while ensuring fast convergence.

Our analysis has two limitations. The first is the continuous approximation of the discontinuous two-phase algorithm, with a three-phase algorithm; since Theorem 8 is independent of the parameter δ, it appears that this approximation has reasonable motivation. Also, later, we will show through simulations that this limitation does not appear to be significant. The second limitation is that we assume $\chi_i < \hat{x}_i$. We believe that this is not a serious limitation since we only need a rough estimate of $\hat{x}_i$ to choose χ_i. However, it would certainly be better if no knowledge of $\hat{x}_i$ is required at all. Despite these limitations, we believe that the proof technique here is an important first step towards analyzing slow-start. In particular, it demonstrates the fact that, if the slow-start phase ends before the equilibrium point is reached, then global stability is preserved.

VI. SIMULATION

Thus far in the paper, we have provided a sufficient condition for global stability. This section is devoted to further investigations of our algorithm, but now using simulations. First, we will consider the rate of convergence. Our global stability condition imposes no constraint on the gains κ_i, but the question arises as to whether there are preferred values for the gains. From our simulations, we will see that the values κ_i can be chosen judiciously to make the network converge quickly.

The second thing we will consider is the impact of window flow control using NS2 (Network Simulator). Our stability condition is based on a rate-based model, but TCP implementations are window-based. Using NS2, we will explore how our derived condition performs in a window-based example. Accordingly, this section will be divided into two parts. In the first part, we will implement the TCP-like algorithm using Matlab to analyze rate of convergence and will also implement the two-phase algorithm and compare it with the single-phase case. In the second part, we will use NS2 to implement window flow control and observe the performance of the network.

A. Rate-Based Simulation

In this subsection, we use Matlab to implement the rate-based algorithm. We use Matlab simulations to study two problems. First, we study the relationship between the rate of convergence and different choices of κ_i; then, we implement the two-phase algorithm to see how it helps the rate of convergence.

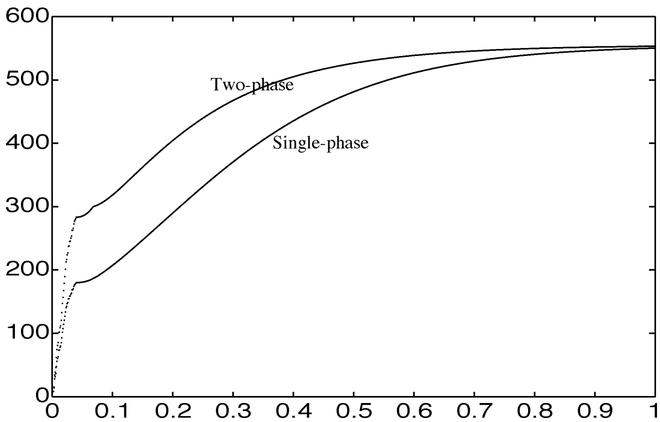


Fig. 6. Two-phase algorithm versus single-phase algorithm.

The network topology we use is that shown in Fig. 4, which has five 1-Gb/s links with 10-ms delay, and there are four routes with 50 users on each route. The network parameters used in the simulations are:

1) propagation delay on each link is 10 ms;
2) capacity of each link is 1 Gb/s;
3) there are 50 users on every route;
4) the route-link incidence matrix R ($R_{il}=1$ indicates that route i uses link l) is

$$R=\begin{pmatrix}1&1&0&0\\0&1&0&1\\0&1&1&0\\1&0&0&1\\0&0&1&1\end{pmatrix}.$$

We use the congestion control algorithm

$$\dot{x}_i(t)=\kappa_i x_i(t-T_i)\left(\frac{100}{x_i^n(t)}-q_i(t)x_i^m(t)\right)$$

and

$$p_l(t)=\left(\frac{y_l(t)}{c_l}\right)^h$$

where $n=0.1$, $m=h=1$, and $c_l=10^6$. The initial conditions (before time zero) of the individual flows are uniformly distributed in the interval [0,2000].

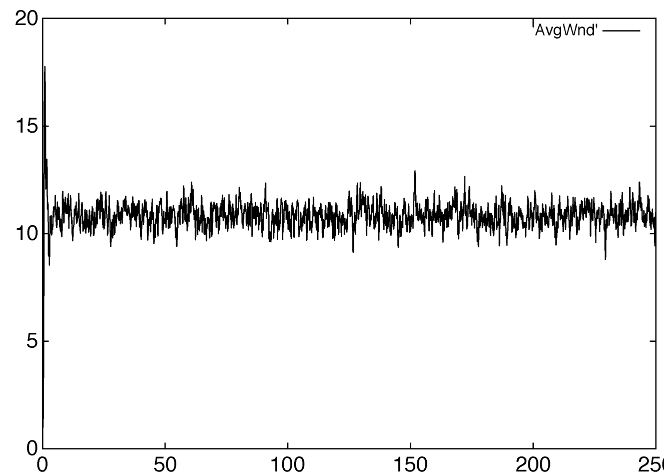
1) Stability and Rate of Convergence: In Section III, we have shown that, if $m_i+n_i>H_i$, the network is globally stable for any $\kappa_i>0$. However, the choice of κ_i may still influence the transient performance of the congestion control algorithm. In this subsection, we will study via simulation the effect of the gains κ_i on system performance. We run the simulations for three different values κ_i: $\kappa_i=0.1$, $\kappa_i=1$, and $\kappa_i=5$. The discrete-time Matlab algorithm we use is shown in Table I. Here, we use $\text{step}=0.5$ ms and run the simulation for 1 s.

The simulation results are shown in Fig. 5. We notice that, when $\kappa_1=0.1$, the rate of convergence is quite slow; when $\kappa_1=5$, the oscillation is quite big. So, the best choice among these three values of κ_1 seems to be $\kappa_1=1$. For this value of κ_1, the rate allocated to user 1 converges fast and the oscillation is small. This suggests that, even though the network may be stable for many values of $\{\kappa_i\}$, these parameters have to be chosen carefully to provide a tradeoff between transient performance and the rate at which the equilibrium is reached. We do not have a prescription for the choice of κ_i in this paper, but it is an important area for further study.

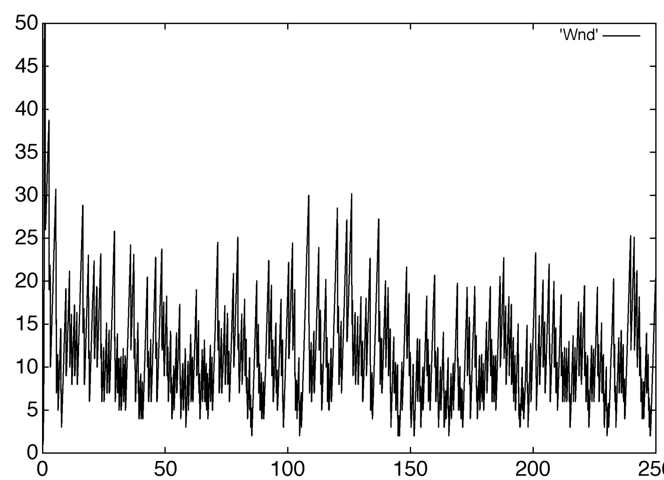
2) Two-Phase Algorithm: In Section V, we have shown that, under the two-phase algorithm, the network is globally, asymptotically stable. Here, we implement the two-phase algorithm (24) and compare it with the single-phase algorithm. To implement the two-phase algorithm, we modify each user's rate update as shown in Table II. For $\kappa_i=0.1$ and $l_i=-0.1$, the

TABLE III
PARAMETERS USED IN THE NS2 SIMULATIONS WITH USERS HAVING IDENTICAL RTT

	h	c	$\hat{w}_i$
Case 1	1.9	5625	11.9252
Case 2	10	973.5	12.1877
Case 3	50	625	11.4950



(a)



(b)

Fig. 7. NS2 simulation with $h=1.9$ and 50 TCP connections starting at the same time. (a) Average window size versus time $(h=1.9)$. (b) Window size of a typical user versus time $(h=1.9)$.

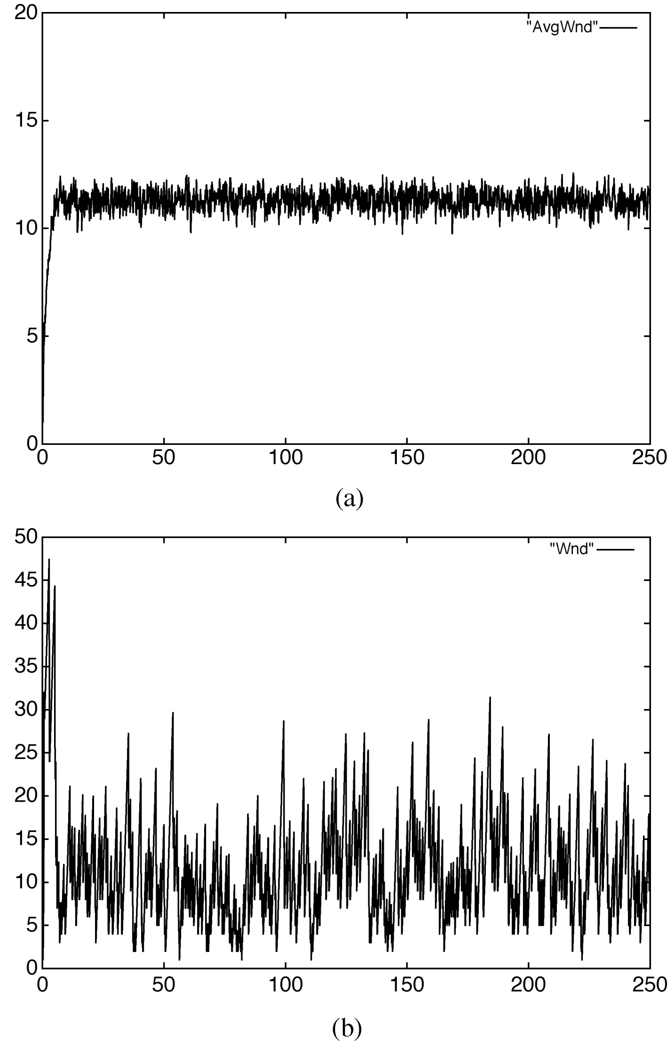


Fig. 8. NS2 simulation with $h = 10$ and 50 TCP connections starting at the same time. (a) Average window size versus time $(h = 10)$. (b) Window size of a typical user versus time $(h = 10)$.

TABLE IV
MEAN AND STANDARD DEVIATIONS IN THE TIME INTERVAL [100,250]

	Average Window Size		Window Size of a Typical User	
	Mean	Standard Deviation	Mean	Standard Deviation
Stable	10.8231	0.515054	11.1262	5.19201
Local Stable	11.284	0.444837	11.7671	5.44449
Unstable	7.44695	0.505862	6.19182	6.25733

results are shown in Fig. 6. We can see that the two-phase algorithm converges faster than the single-phase one.

Note here that our implementation is a discrete-time version of the two-phase algorithm given in (24) and not the continuous approximation in (26). Our simulation shows that (24) might be stable even though our analysis works only for (26).

B. NS2 Simulations

In the following simulations, NS2 is used to implement TCP-Reno, which corresponds to $m_i = 1, n_i = 1$ [28], [34]. We only consider the single-link case and the link capacity is taken to be 50 Mb/s, with a one-way propagation delay of 50 ms. REM [3]

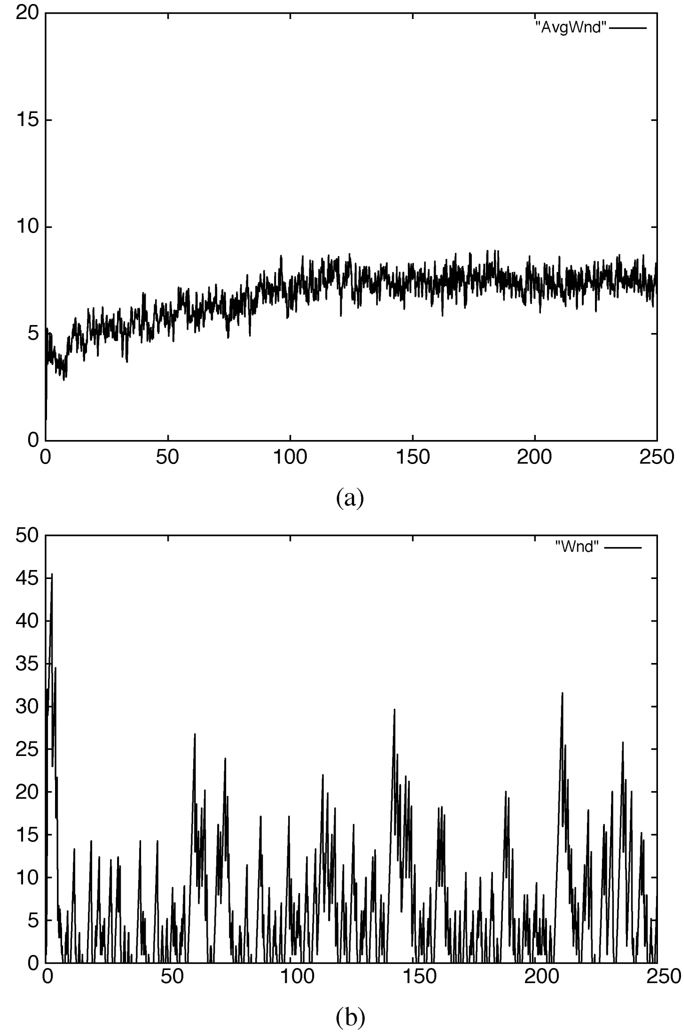


Fig. 9. NS2 simulation with $h = 50$ and 50 TCP connections starting at the same time. (a) Average window size versus time $(h = 50)$. (b) Window size of a typical user versus time $(h = 50)$.

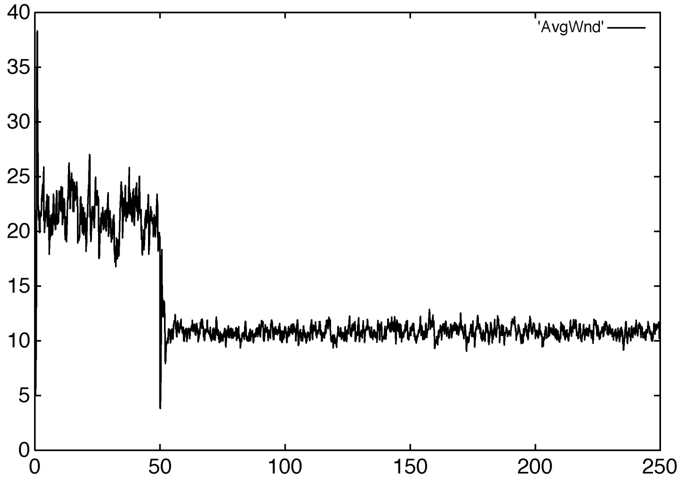
is enabled so that we can estimate the link price at the sources. The price function we use is

$$f(y) = \left(\frac{y}{c}\right)^h$$

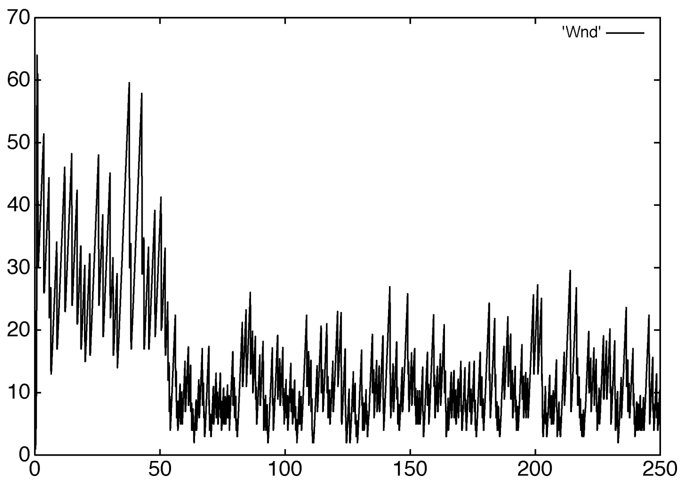
where h is varied in the simulations to study the impact of the global stability condition. The parameter c is adjusted in each simulation to make the equilibrium arrival rate at the link equal to the link capacity so that the different cases can be compared while keep the target utilization equal to one. We study the performance of 50 TCP connections on this link.

1) Persistent TCP Connections: In the first set of simulations, we let all 50 TCP connections start at the same time and last for the entire duration of the simulation. We consider three different sets of parameters given in Table III.

Here, Case 1 satisfies the global stability condition. In the second case, the system is locally stable but does not satisfy our global stability condition. In the third case, we can use the local analysis to show that the system is unstable. In this simulation, all connections start at time 0 and finish after 250 s. The simulation results are shown in Figs. 7–9.

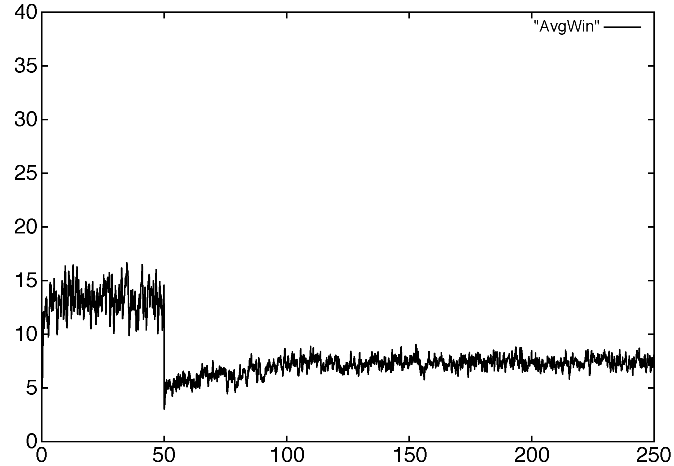


(a)

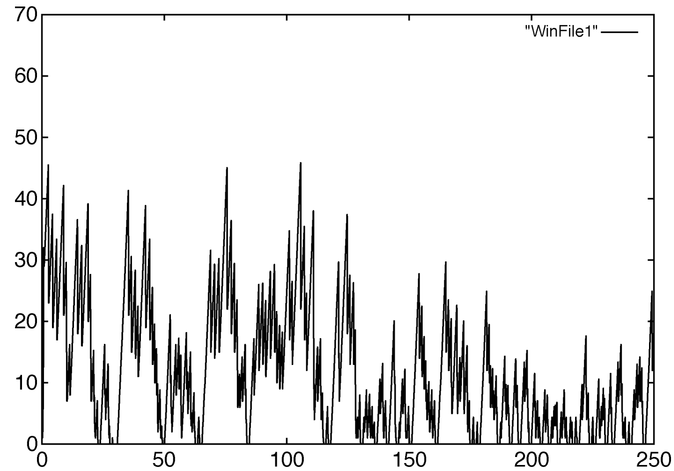


(b)

Fig. 10. NS2 simulation with $h = 1.9$ and 50 TCP connections starting at different times. (a) Average window size versus time $(h = 1.9)$. (b) Window size of a typical user versus time $(h = 1.9)$.



(a)



(b)

Fig. 11. NS2 simulation with $h = 50$ and 50 TCP connections starting at different times. (a) Average window size versus time $(h = 50)$. (b) Window size of a typical user versus time $(h = 50)$.

We calculate the mean and the standard deviation in the time interval [100,250] of both the average window size and the window size of an individual user in Table IV. First, we compare the stable case with the unstable case. We can see that the standard deviations of the average window size are almost the same. But, in the stable case, $\bar{w} = 10.8231 \approx 11.9252 = \hat{w}$ and, in the unstable case, $\bar{w} = 7.44695 < 11.4950 = \hat{w}$. For individual users, we can see that, for the stable case $\bar{w} = 11.1262 \approx 11.9252$, but in the unstable case $\bar{w} = 6.1982 < 11.4950$.

Now, we compare the locally stable case with the stable case. For a network that satisfies the local stability condition but does not satisfy our global stability condition, the global stability condition is still an open problem. From our simulations, we can see that the stable case and local stable case have similar average window size and standard deviation. This means that even though our global stability condition does not hold, it is possible that the region of attraction, i.e., the set of initial conditions which ensure stability, may be large.

2) Simulations With Sudden Load Change: In the second simulation, we only consider the stable case and the unstable case. We start with 10 TCP connections first for 50 s and then add the other 40 connections to study the transient behavior of the network. For 10 TCP connections, the equilibrium window size will be 26.121 for $h = 1.9$ and 54.0252 for $h = 50$. The simulation results are shown in Figs. 10 and 11.

From Figs. 10(a) and 11(a), first we can see that the users in case 1 achieve the new equilibrium faster than the users in case 2 do, which is consistent with Fig. 6.

VII. Conclusion

An important open problem in the study of Internet congestion control has been the design of congestion controllers which are globally, asymptotically stable in a network with heterogeneous feedback delays. In this paper, we have established the global stability of a class of congestion management algorithms, i.e., a combination of congestion controllers at the sources and congestion signaling mechanisms at the routers. The proof of global stability is obtained by placing certain restrictions on the increase/decrease parameters of TCP and the parameters of the link price functions. A similar result for a slightly different class of single-phase controllers has been obtained in [17] independently using different techniques. A significant open question for further research is the following: for congestion management algorithms that are outside the class considered in this paper, is it possible to stabilize the system using a *slow-start* procedure to bring the system close to its equilibrium and choosing the control parameters to ensure local stability?

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